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JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 31, 509–516 (1970)

Boundedness Theorems for Certain Second Order Nonlinear Differential Equations

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1. INTRODUCTION

In this paper we shall study the boundedness of solutions of the following second order nonlinear differential equations:

$$(r(t)\, u')' + f(t, u)\, g(u') = q(t), \qquad \left(' = \frac{d}{dt}\right), \tag{1}$$

and

$$(r(t)\, u')' + f(t, u)\, g(u') = 0, \tag{2}$$

where $r(t)$ is positive and absolutely continuous over the half infinite interval $t \geqslant 0$, $f(t, x)$ is continuous for all $t \geqslant 0$ and for all values of x, $xf(t, x) > 0$ for $x \neq 0$ and for all $t \geqslant 0$, $g(x)$ is continuous and positive for all x, and for equation (1) the forcing term $q \in \mathscr{L}_1(0, \infty)$.

Equations of these types have been previously studied by Lalli [2], where he took $f(t, u) = a(t) f(u)$. Wong [6] discussed equation (2) by taking $r(t) \equiv 1$ and $g(u') \equiv 1$. Wong and Burton [7] studied equation (2) by taking $r(t) \equiv 1$ and $f(t, u) = a(t) f(u)$. In case $g(u') \equiv 1$ and $f(t, u) = a(t) f(u)$, equation (2) has been studied by Wong [5]. Waltman [4] discussed equation (2) by taking $r(t) \equiv 1$, $g(u') \equiv 1$ and $f(t, u) = a(t) f(u)$, and generalized the corresponding result for the linear case (Bellman [1], p. 113).

Boundedness theorems for equations (1) and (2) are given in Section 2. For the case $f(t, u) = a(t) f(u)$, we present certain results on boundedness and stability in Section 3, which generalize and complement the results given in [2]. The following fundamental lemma (Bellman [1], p. 35) will be needed.

LEMMA. *If $u, v \geqslant 0$, if c is a positive constant, and if*

$$u \leqslant c + \int_0^t uv\, ds$$

then

$$u \leqslant c \exp\left(\int_0^t v\, ds\right).$$

2. Boundedness Theorems

Theorem 1. *If we assume that*

(i) $\int_0^v \frac{x}{g(x)}\, dx \to \infty$ *as* $|v| \to \infty$,

(ii) *there exist nonnegative constants* k_1 *and* k_2 *such that*

$$\frac{|y|}{g(y)} \leqslant k_1 + k_2 \int_0^y \frac{x}{g(x)}\, dx,$$

(iii) *there exists a nonnegative function* $\beta \in \mathscr{L}_1(0, \infty)$ *such that*

$$\frac{f_t(t, x)}{f(t, x)} \leqslant \beta(t) \qquad \text{for all } x,$$

(iv) *there exist a nonnegative function* $p(x)$ *and constants* $x_0 \geqslant 0$ *and* $T \geqslant 0$ *such that* $|f(t, x)| \geqslant p(x)$ *for all* $t \geqslant T$ *and all* $|x| \geqslant x_0$ *and*

$$\int_0^\infty p(x)\, dx = \int_{-\infty}^0 p(x)\, dx = \infty,$$

(v) $r'(t) \geqslant 0$ *and* $\lim_{t\to\infty} r(t) = B$ *for some positive constant* B, *then for each solution* $u(t)$ *of equation* (1) *both* $|u(t)|$ *and* $|u'(t)|$ *are bounded.*

Proof. For any solution $u(t)$ of (1) we define

$$W(t) = r(t) \int_0^{u'(t)} \frac{x}{g(x)}\, dx + \int_0^{u(t)} f(t, x)\, dx, \tag{3}$$

from which it follows by (1) that

$$W'(t) = \frac{q(t)\, u'}{g(u')} - \frac{r'(t)\, u'^2}{g(u')} + r'(t) \int_0^{u'(t)} \frac{x}{g(x)}\, dx + \int_0^{u(t)} f_t(t, x)\, dx. \tag{4}$$

By the assumptions (ii), (iii), and (v), it can be shown easily that

$$\begin{aligned}
W'(t) &\leqslant |q(t)| \left(k_1 + k_2 \int_0^{u'(t)} \frac{x}{g(x)}\,dx\right) + r'(t) \int_0^{u'(t)} \frac{x}{g(x)}\,dx \\
&\quad + \beta(t) \int_0^{u(t)} f(t, x)\,dx \\
&\leqslant k_1 |q(t)| + \frac{k_2}{r(t)} |q(t)|\, W(t) + \frac{r'(t)}{r(t)} W(t) + \beta(t)\, W(t) \\
&\leqslant k_1 |q(t)| + \left(\frac{k_2}{r(0)} |q(t)| + \frac{r'(t)}{r(t)} + \beta(t)\right) W(t).
\end{aligned}$$

Integrating from 0 to some $t \geqslant 0$, we obtain

$$W(t) - W(0) \leqslant k_1 \int_0^t |q(s)|\,ds + \int_0^t \left(\frac{k_2}{r(0)} |q(s)| + \frac{r'(s)}{r(s)} + \beta(s)\right) W(s)\,ds.$$

Since $q \in \mathscr{L}_1(0, \infty)$, this implies that

$$W(t) \leqslant C_1 + \int_0^t \left(\frac{k_2}{r(0)} |q(s)| + \frac{r'(s)}{r(s)} + \beta(s)\right) W(s)\,ds, \qquad C_1 > 0.$$

Using lemma, we get

$$\begin{aligned}
W(t) &\leqslant C_1 \exp\left(\int_0^t \left(\frac{k_2}{r(0)} |q(s)| + \frac{r'(s)}{r(s)} + \beta(s)\right) ds\right) \\
&\leqslant C_2
\end{aligned}$$

for some constant $C_2 > 0$. Then it follows that

$$r(t) \int_0^{u'(t)} \frac{x}{g(x)}\,dx \leqslant C_2\,, \tag{5}$$

and

$$\int_0^{u(t)} f(t, x)\,dx \leqslant C_2. \tag{6}$$

Using the assumption (i) and the fact that $r(t) \geqslant r(0) > 0$ for all $t \geqslant 0$, we conclude from (5) that $|u'(t)|$ is bounded. By the assumption (iv), (6) implies that $|u(t)|$ is bounded. For if not, we may assume without loss of generality that there exists a sequence $\{t_n\}$ such that $u(t_n) \to \infty$ as $t_n \to \infty$. Then for all t_n sufficiently large we have by (iv) that

$$\int_{x_0}^{u(t_n)} f(t_n\,, x)\,dx \geqslant \int_{x_0}^{u(t_n)} p(x)\,dx.$$

This is clearly a contradiction to (6) since the right side tends to ∞ as $t_n \to \infty$. The proof is thus complete.

Remarks. (1) When $g(x) \equiv 1$, the conditions (i) and (ii) are always satisfied. For (ii), we may take $k_1 = \frac{1}{2}$ and $k_2 = 1$.

(2) If $r(t) \equiv 1$, $g(u') \equiv 1$ and $q(t) \equiv 0$, then Theorem 1 reduces to the result given in [6] (Theorem 7, p. 222).

It is clear from the proof of Theorem 1 that if $q(t) \equiv 0$ then the condition (ii) can be disregarded. We have the following immediate corollary.

COROLLARY 1. *If we have the conditions* (i), (iii), (iv) *and* (v), *then for each solution* $u(t)$ *of* (2) *both* $|u(t)|$ *and* $|u'(t)|$ *are bounded.*

THEOREM 2. *If we assume the conditions* (i)-(iv) *as in Theorem* 1 *and that*

(vi) $r'(t) \leqslant 0$ *and* $\lim_{t\to\infty} r(t) = b > 0$,

(vii) $x^2/g(x) \leqslant K$ *for some* $K > 0$ *and for all* x,

then for each solution $u(t)$ *of* (1) *both* $|u(t)|$ *and* $|u'(t)|$ *are bounded.*

Proof. We define $W(t)$ as in (3). Using the conditions (ii), (iii), (vi), and (vii), we obtain from (4) that

$$\begin{aligned} W'(t) &\leqslant |q(t)| \left(k_1 + k_2 \int_0^{u'(t)} \frac{x}{g(x)}\, dx\right) + K\,|r'(t)| + \beta(t) \int_0^{u(t)} f(t, x)\, dx \\ &\leqslant k_1\,|q(t)| + K\,|r'(t)| + \left(\frac{k_2}{b}\,|q(t)| + \beta(t)\right) W(t). \end{aligned}$$

Note that condition (vi) implies that $r' \in \mathscr{L}_1(0, \infty)$. Now, integrating from 0 to $t \geqslant 0$, we get

$$W(t) \leqslant C_1 + \int_0^t \left(\frac{k_2}{b}\,|q(s)| + \beta(s)\right) W(s)\, ds, \qquad C_1 > 0.$$

Thus we obtain by lemma that

$$\begin{aligned} W(t) &\leqslant C_1 \exp\left(\int_0^t \left(\frac{k_2}{b}\,|q(s)| + \beta(s)\right) ds\right) \\ &\leqslant C_2, \end{aligned}$$

for some constant $C_2 > 0$. Similarly as in Theorem 1, we conclude that both $|u(t)|$ and $|u'(t)|$ are bounded.

Remark. An example of function which satisfies all the conditions (i), (ii), and (vii) is that $g(x) = x^2 + 1$. For (ii), we may simply take $k_1 = \frac{1}{2}$ and $k_2 = 0$.

COROLLARY 2. *If we have the conditions* (i), (iii), (iv), (vi) *and* (vii), *then for each solution* $u(t)$ *of* (2) *both* $|u(t)|$ *and* $|u'(t)|$ *are bounded.*

In the previous theorems, we have assumed the monotonicity of $r(t)$. When $g(u') \equiv 1$, this condition may be relaxed. We have the following theorem for the equation

$$(r(t)\, u')' + f(t, u) = q(t), \tag{7}$$

where r, f, and q are the same as in equation (1).

THEOREM 3. *If we assume the conditions* (iii), (iv) *and that* (viii) $r(t) \geqslant b > 0$ *for all* $t \geqslant 0$, *and* $\int_0^\infty [r_-'(t)/r(t)]\, dt < \infty$, *then for each solution* $u(t)$ *of* (7) *both* $|u(t)|$ *and* $|u'(t)|$ *are bounded.*

Proof. For any solution $u(t)$ of (7) we define

$$W(t) = r(t)\frac{u'^2(t)}{2} + \int_0^{u(t)} f(t, x)\, dx.$$

Calculating the derivative of $W(t)$ and applying the conditions (iii) and (viii), one can show that

$$W'(t) \leqslant \frac{1}{2}|q(t)| + \left(\frac{1}{b}|q(t)| + \frac{r_-'(t)}{r(t)} + \beta(t)\right) W(t).$$

Integrating form 0 to $t \geqslant 0$ and using lemma, we conclude that $W(t)$ is bounded. This completes the proof.

3. THE CASE $f(t, u) = a(t)f(u)$

In this section we consider the differential equations:

$$(r(t)\, u')' + a(t) f(u)\, g(u') = q(t) \tag{8}$$

and

$$(r(t)\, u')' + a(t) f(u)\, g(u') = 0, \tag{9}$$

where $a(t)$ is positive and absolutely continuous over the half infinite interval $t \geqslant 0, f(x)$ is continuous for all x, $xf(x) > 0$ for $x \neq 0$, and the functions r, g and q are the same as in equation (1).

THEOREM 4. *If in addition to the conditions* (i), (ii), *and* (v) *we assume that*

(ix) $\int_0^u f(x)\,dx \to \infty$ *as* $|u| \to \infty$,

(x) $a(t) \geqslant \alpha > 0$ *for all* $t \geqslant 0$, *and*

$$\int_0^\infty \frac{a_+'(t)}{a(t)}\,dt < \infty,$$

then for each solution $u(t)$ *of* (8) *both* $|u(t)|$ *and* $|u'(t)|$ *are bounded.*

Proof. For any solution $u(t)$ of (8) we define

$$V(t) = r(t) \int_0^{u'(t)} \frac{x}{g(x)}\,dx + a(t) \int_0^{u(t)} f(x)\,dx. \tag{10}$$

Then we find that

$$V'(t) \leqslant k_1 |q(t)| + \left(\frac{k_2}{r(0)} |q(t)| + \frac{r'(t)}{r(t)} + \frac{a_+'(t)}{a(t)}\right) V(t).$$

Similarly as in Theorem 1, we obtain the desired conclusion.

Remark. The above theorem contains the result given by Lalli [2] (Theorem 5, p. 187), where he assumed that $a(t)$ is nonincreasing and g satisfies the inequality:

$$\frac{|y|}{r(0)\,g(y)} \leqslant 1 + \int_0^y \frac{x}{g(x)}\,dx,$$

along with other suitable conditions.

COROLLARY 3. *If we have the conditions* (i), (v), (ix), *and* (x), *then for each solution* $u(t)$ *of* (9) *both* $|u(t)|$ *and* $|u'(t)|$ *are bounded.*

COROLLARY 4. *If in Corollary* 3 *the condition* (x) *is replaced by* (x)$'$ $a(t) \leqslant A$ *for all* $t \geqslant 0$ *and for some constant* $A > 0$, *and*

$$\int_0^\infty \frac{a_-'(t)}{a(t)}\,dt < \infty,$$

then the same conclusion holds.

The proof of this corollary is easy, and hence omitted.

Remark. When $r(t) \equiv 1$, Corollary 3 and 4 include the corresponding results in [7].

The following theorem allows that $r'(t) \leqslant 0$.

THEOREM 5. *If we assume the conditions* (i), (ii), (vi), (vii), (ix), *and* (x), *then for each solution* $u(t)$ *of* (8) *both* $|u(t)|$ *and* $|u'(t)|$ *are bounded.*

Proof. For any solution $u(t)$ of (8) we define $V(t)$ as in (10). Then

$$V'(t) \leqslant k_1 |q(t)| + K |r'(t)| + \left(\frac{k_2}{b} |q(t)| + \frac{a_+'(t)}{a(t)}\right) V(t).$$

The rest of the proof is similar to that of Theorem 2.

COROLLARY 5. *If we have the conditions* (i), (vi), (vii), (ix) *and* (x), *then for each solution* $u(t)$ *of* (9) *both* $|u(t)|$ *and* $|u'(t)|$ *are bounded.*

Remark. Corollary 5 includes both Theorem 2 and Theorem 3 in [2].

Let $x(t) = u(t)$ and $y(t) = u'(t)$. We note that equation (9) is equivalent to the system:

$$\begin{aligned} x' &= y \\ y' &= -\frac{r'(t)}{r(t)} y - \frac{a(t)}{r(t)} f(x) g(y). \end{aligned} \tag{11}$$

We have the following stability theorems for the system (11).

THEOREM 6. *If we assume the condition* (v) *and that*

(xi) $a(t) \geqslant \alpha > 0$ *for all* $t \geqslant 0$, *and* $r(t) a'(t) - r'(t) a(t) \leqslant 0$ *for all* $t \geqslant 0$,

then the solution $(x(t), y(t)) \equiv 0$ *of the system* (11) *is stable in the sense of Liapunov* (*for terminology see* [3]).

Proof. The function defined by

$$V(x, y, t) = \int_0^y \frac{\sigma}{g(\sigma)} d\sigma + \frac{a(t)}{r(t)} \int_0^x f(s)\, ds$$

is a Liapunov function for the system (11).

Remark. Theorem 6 contains both Theorem 1 and its Corollary in [2], where the monotonicity of $a(t)$ is assumed.

THEOREM 7. *If in Theorem* 6 *the conditions* (v) *and* (xi) *are replaced by*

(v)′ $r'(t) \geqslant 0$, *and*

(xi)′ $a(t) \leqslant A$ *for all* $t \geqslant 0$ *and for some constant* $A > 0$, *and*

$$r'(t) a(t) - r(t) a'(t) \leqslant 0 \qquad \text{for all} \qquad t \geqslant 0,$$

then the same conclusion holds.

Proof. The function defined by

$$V(x, y, t) = \frac{r(t)}{a(t)} \int_0^y \frac{\sigma}{g(\sigma)}\, d\sigma + \int_0^x f(s)\, ds$$

is now a Liapunov function for the system (11).

When $g(u') \equiv 1$, the monotonicity assumption regarding $r(t)$ in Theorem 4 and 5 may be relaxed. Consider now the equation

$$(r(t)\, u')' + a(t) f(u) = q(t), \tag{12}$$

where the functions r, a, f, and q are the same as in equation (8). With obvious modification of the method of proof used there, one can obtain immediately the following results.

THEOREM 8. *If we assume the conditions* (viii), (ix), *and* (x), *then for each solution* $u(t)$ *of* (12) *both* $|u(t)|$ *and* $|u'(t)|$ *are bounded.*

THEOREM 9. *If in addition to the conditions* (viii) *and* (ix) *we assume that*

(xii) $a(t) \geqslant \alpha > 0$ *for all* $t \geqslant 0$, *and*

$$\int_0^\infty \frac{a_-'(t)}{a(t)}\, dt < \infty,$$

then all solutions of (12) *are bounded.*

Remark. In Theorem 9, if $q(t) \equiv 0$ then the requirements $a(t) \geqslant \alpha > 0$ and $r(t) \geqslant b > 0$ can be disregarded. In that case, Theorem 9 reduces to the result given in [5].

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